Study materials on Magnetostatics – Part II

SEM – II   Paper - CC-III
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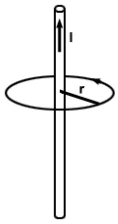
Vector Properties of the Magnetic Field

Line Integrals of Magnetic Fields:

Recall that while studying electric fields we established that the surface integral through any closed surface in the field was equal to 4Π times the total charge enclosed by the surface. We wish to develop a similar property for magnetic fields. For magnetic fields, however, we do not use a closed surface, but a closed loop. Consider a closed circular loop of radius R about a straight wire carrying a current I, as shown below.



A closed path around a straight wire

What is the line integral around this closed loop? We have chosen a path with constant radius, so the magnetic field at every point on the path is the same: B = . In addition, the total length of the path is simply the circumference of the circle: L = 2Πr. Thus, because the field is constant on the path, the line integral is simply:

Line integral $\int \boldsymbol{B.dS} = \boldsymbol{Bl} = \frac{\boldsymbol{2I}}{\boldsymbol{rc}}\,(\boldsymbol{2\pi r}) = \frac{\boldsymbol{4\pi I}}{\boldsymbol{c}}$

This equation, called Ampere's Law, is quite convenient. We have generated an equation for the line integral of the magnetic field, independent of the position relative to the source. In fact, this equation is valid for any closed loop around the wire, not just a circular one.

$$\int \boldsymbol{B.dS} = \frac{\boldsymbol{4\pi}}{\boldsymbol{c}}\ \textbf{x total current enclosed by the path}$$

Note that the path need not be circular or perpendicular to the wires. The figure below shows a configuration of a closed path around a number of wires:

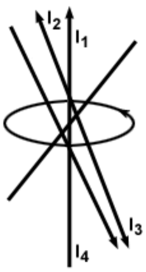


Figure: A closed path enclosing 4 wires

The line integral around the circle in the figure is equal to $\frac{4\pi}{c}(I_1 + I_2 - I_3 - I_4)$. Notice that the two wires pointing downwards are subtracted, since their field points in the opposite direction from the curve.

This equation, similar to the surface integral equation for electric fields, is powerful and allows us to greatly simplify many physical situations.

The Curl of a Magnetic Field:

From this equation, we can generate an expression for the curl of a magnetic field. Stokes' Theorem states that:

$$\int \boldsymbol{B.dS} = \int \boldsymbol{curl\,B.da}$$

We have already established that

$$\int \boldsymbol{B.dS} = \frac{\boldsymbol{4\pi I}}{\boldsymbol{c}}$$

Thus:

$$\int \boldsymbol{curl\,B.da} = \frac{\boldsymbol{4\pi I}}{\boldsymbol{c}}$$

To remove the integral from this equation we include the concept of current density J. Recall that $I = \int J.da$

Substituting this into our equation, we find that

$$\int \boldsymbol{curl\,B.da} = \frac{\boldsymbol{4\pi}}{\boldsymbol{c}} \int \boldsymbol{J.da}$$

Clearly, then:

$$\boldsymbol{curl\,B} = \frac{\boldsymbol{4\pi J}}{\boldsymbol{c}}$$

Thus, the curl of a magnetic field at any point is equal to the current density at that point. This is the simplest statement relating the magnetic field and moving charges.

The Divergence of the Magnetic Field:

Recall that the divergence of the electric field was equal to the total charge density at a given point. We have already examined qualitatively that there is no such thing as magnetic charge. All magnetic fields are, in essence, created by moving charges, not by static ones. Thus, because there are no magnetic charges, there is no divergence in a magnetic field: **div B = 0.** This fact remains true for any point in any magnetic field. Our expressions for divergence and curl of a magnetic field are sufficient to describe uniquely any magnetic field from the current density in the field. The equations for divergence and curl are extremely powerful; taken together with the equations for the divergence and curl for the electric field, they are said to encompass mathematically the entire study of electricity and magnetism.

The Magnetic Vector Potential:

Electric fields generated by stationary charges obey

$$\nabla \times \mathbf{E} = 0 \qquad \mathbf{(1)}$$

This immediately allows us to write

$$\mathbf{E} = -\nabla\phi \qquad (2)$$

since the curl of a gradient is automatically zero. In fact, whenever we come across an irrotational vector field in physics we can always write it as the gradient of some scalar field. This is clearly a useful thing to do, since it enables us to replace a vector field by a much simpler scalar field. The quantity ϕ in the above equation is known as the electric scalar potential.

Magnetic fields generated by steady currents (and unsteady currents, for that matter) satisfy

$$\nabla \cdot \mathbf{B} = 0 \qquad (3)$$

This immediately allows us to write

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad (4)$$

since the divergence of a curl is automatically zero. In fact, whenever we come across a solenoidal vector field in physics we can always write it as the curl of some other vector field. This is not an obviously useful thing to do, however, since it only allows us to replace one vector field by another. Nevertheless, Eq. (4) is one of the most useful equations we shall come across in this lecture course. The quantity **A** is known as the magnetic vector potential.

The curl of the vector potential gives us the magnetic field via Eq. (318). However, the divergence of **A** has no physical significance. In fact, we are completely free to choose $\nabla\cdot\mathbf{A}$ to be whatever we like. Note that, according to Eq. (4), the magnetic field is invariant under the transformation

$$\mathbf{A} \rightarrow \mathbf{A} - \nabla\psi \qquad (5)$$

In other words, the vector potential is undetermined to the gradient of a scalar field. This is just another way of saying that we are free to choose ∇.A . Recall that the electric scalar potential is undetermined to an arbitrary additive constant, since the transformation

$$\phi \rightarrow \phi + c \qquad (6)$$

leaves the electric field invariant in Eq. (2). The transformations (5) and (6) are examples of what mathematicians call gauge transformations. The choice of a particular function ψ or a particular constant c is referred to as a choice of the gauge. We are free to fix the gauge to be whatever we like. The most sensible choice is the one which makes our equations as simple as possible. The usual gauge for the scalar potential ϕ is such that $\phi \rightarrow 0$ at infinity. The usual gauge for **A** is such that

$$\nabla \cdot \mathbf{A} = 0 \qquad (7)$$

This particular choice is known as the **Coulomb gauge.**

It is obvious that we can always add a constant to ϕ so as to make it zero at infinity. But it is not at all obvious that we can always perform a gauge transformation such as to make $\nabla \times \mathbf{A}$ zero. Suppose that we have found some vector field **A** whose curl gives the magnetic field but whose divergence in non-zero. Let

$$\nabla \cdot \mathbf{A} = \nu(\mathbf{r}) \qquad (8)$$

The question is, can we find a scalar field ψ such that after we perform the gauge transformation (5) we are left with $\nabla \cdot \mathbf{A} = 0$. Taking the divergence of Eq. (5) it is clear that we need to find a function ψ which satisfies

$$\nabla^2\psi = \nu \qquad (9)$$

But this is just Poisson's equation. We know that we can always find a unique solution of this equation. This proves that, in practice, we can always set the divergence of **A** equal to zero.

Biot-Savart's Law for Vector Potential

Biot-Savart's law for magnetic field due to a current element $\vec{dl}$

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{\vec{dl} \times \hat{r}}{r^2} = -\frac{\mu_0 I}{4\pi}\vec{dl} \times \nabla(\frac{1}{r})$$

may be used to obtain an expression for the vector potential. Since the element $\vec{dl}$ does not depend on the position vector of the point at which the magnetic field is calculated, we can write $d\vec{B} = \frac{\mu_0 I}{4\pi}\nabla \times (\frac{\vec{dl}}{r})$

the change in sign is because $\nabla(\frac{\vec{dl}}{r}) = \nabla(1/r) \times \vec{dl}$.

Thus the contribution to the vector potential from the element $\vec{dl}$ is $d\vec{A} = \frac{\mu_0 I}{4\pi r}\vec{dl}$

The expression is to be integrated over the path of the current to get the vector potential for the system

$$\vec{A} = \frac{\mu_0 I}{4\pi}\int \frac{\vec{dl}}{r}$$

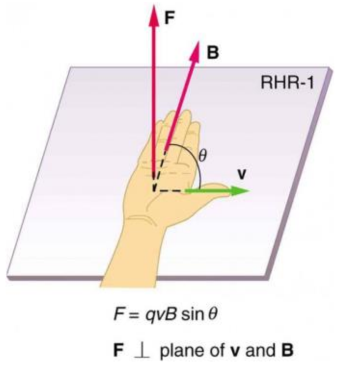
Force on a Moving point Charge in a Magnetic Field:

The magnitude of the magnetic force F on a charge q moving at a speed v in a magnetic field of strength B is given by

$$\mathbf{F = qvB \sin\theta},$$

where θ is the angle between the directions of v and B. This force is often called the Lorentz force. In fact, this is how we define the magnetic field strength B—in terms of the force on a charged particle moving in a magnetic field.

The direction of the magnetic force F is perpendicular to the plane formed by v and B, as determined by the right hand rule 1 (or RHR-1), which is illustrated in Figure 1. RHR-1 states that, to determine the direction of the magnetic force on a positive moving charge, you point the thumb of the right hand in the direction of v, the fingers in the direction of B, and a perpendicular to the palm points in the direction of F. One way to remember this is that there is one velocity, and so the thumb represents it. There are many field lines, and so the fingers represent them. The force is in the direction you would push with your palm. The force on a negative charge is in exactly the opposite direction to that on a positive charge.



Magnetic Force on a Current-Carrying Conductor:

A current-carrying wire in a magnetic field must therefore experience a force due to the field. To investigate this force, let's consider the infinitesimal section of wire as shown in the above Figure. The length and cross-sectional area of the section are dl and A, respectively, so its volume is **V=A· dl**. The wire is formed from material that contains n charge carriers per unit volume, so the number of charge carriers in the section is nA· dl. If the charge carriers move with drift velocity $\vec{v_d}$ the current I in the wire is (from Current and Resistance)

$$\mathbf{I = neAv_d} \qquad (1)$$

The magnetic force on any single charge carrier is $e\vec{v_d}\times\vec{B}$, so the total magnetic force $d\vec{F}$ on the nA·dl charge carriers in the section of wire is

$$d\vec{F} = (nA\cdot dl)\, e\vec{v_d} \times \vec{B} \qquad (2)$$

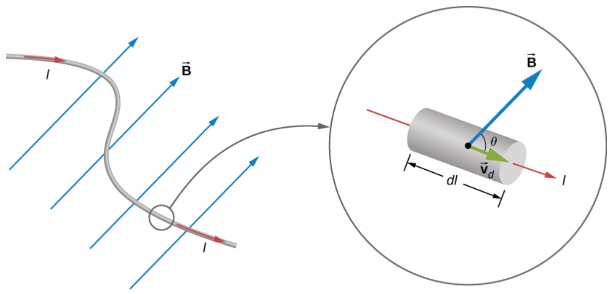
We can define dl to be a vector of length dl pointing along $\vec{v_d}$, which allows us to rewrite this equation as

$$d\vec{F} = neAv_d\vec{dl} \times \vec{B} \qquad (3)$$

or

$$d\vec{F} = I\,\vec{dl} \times \vec{B} \qquad (4)$$

This is the magnetic force on the section of wire. Note that it is actually the net force exerted by the field on the charge carriers themselves. The direction of this force is given by RHR-1, where you point your fingers in the direction of the current and curl them toward the field. Your thumb then points in the direction of the force.



To determine the magnetic force $\vec{F}$ on a wire of arbitrary length and shape, we must integrate Equation (4) over the entire wire. If the wire section happens to be straight and B is uniform, the equation differentials become absolute quantities, giving us

$$\vec{F} = I\vec{l} \times \vec{B} \qquad (5)$$

This is the force on a straight, current-carrying wire in a uniform magnetic field.

Magnetic Force between Two Parallel Currents:

When two parallel, current carrying wires are placed at some distance r from each other, they will experience a force on each other, due to magnetic field produced by each other.

Let us consider the field produced by wire 1 and the force it exerts on wire 2 (call the force F_2). The field due to I_1 at a distance r is given to be

$$\mathbf{B_1} = \frac{\mu_0 I_1}{2\pi r}$$

This field is uniform along wire 2 and perpendicular to it, and so the force F_2 it exerts on wire 2 is given by **F=I*l*Bsinθ** (from Eqn 5) with $\sin\theta = 1$

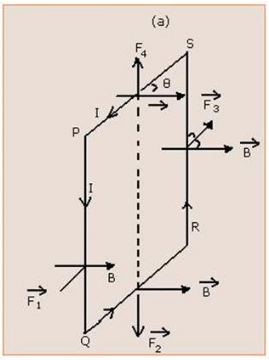
$$F_2 = I_2 l B_1$$

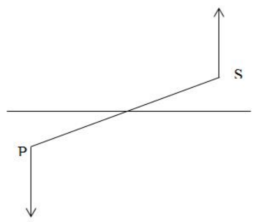
By Newton's third law, the forces on the wires are equal in magnitude, and so we just write F for the magnitude of F_2. (Note that $F_1 = -F_2$.) Since the wires are very long, it is convenient to think in terms of F/*l*, the force per unit length. Substituting the expression for B_1 into the last equation and rearranging terms gives

$$\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi r}$$

F/*l* is the force per unit length between two parallel currents I_1 and I_2 separated by a distance r. The force is attractive if the currents are in the same direction and repulsive if they are in opposite directions. This force is responsible for the pinch effect in electric arcs and plasmas. The force exists whether the currents are in wires or not. In an electric arc, where currents are moving parallel to one another, there is an attraction that squeezes currents into a smaller tube. In large circuit breakers, like those used in neighbourhood power distribution systems, the pinch effect can concentrate an arc between plates of a switch trying to break a large current, burn holes, and even ignite the equipment.

Torque on rectangular coil (current loop) in a magnetic field:





As the current carrying conductor experiences a force when placed in a magnetic field, each side of a current carrying rectangular coil experiences a force in a magnetic field. In the present section we shall see in what way the rectangular loop carrying current is influenced by a magnetic field.

Consider a rectangular coil of length l and breadth b carrying a current I placed in a uniform magnetic field B. θ be the angle between plane of rectangular coil and magnetic field. The magnitude of experienced by each side of loop is given below.

- Force acting on side PQ (F_1):

$$F_1 = I(\overrightarrow{PQ} \times \vec{B})$$
$$F_1 = I\ PQ.B\ sin\theta = I\ l\ B\ sin90$$
$$since\ angle\ between\ side\ PQ\ and\ magnetic\ field\ is\ 90$$
$$F_1 = I\ l\ B$$

the direction of force is perpendicular to the plane containing PQ and B. It is directed outward direction as shown in figure.

- Force acting on side QR (F_2):

$$F_2 = I(\overrightarrow{QR} \times \vec{B})$$
$$F_2 = I\ PQ.B\ sin\theta$$
$$F_2 = I\ b\ B\ sin\theta$$

the direction of force is perpendicular to the plane containing QR and B. It is directed downward direction as shown in figure.

- Force acting on side RS (F_3):

$$F_3 = I(\overrightarrow{RS} \times \vec{B})$$
$$F_3 = I\ RS.B\ sin\theta = I\ l\ B\ sin90$$
$$since\ angle\ between\ side\ PQ\ and\ magnetic\ field\ is\ 90$$
$$F_3 = I\ l\ B$$

  the direction of force is perpendicular to the plane containing RS and B. It is directed inward direction as shown in figure.

- Force acting on side SP (F_4):

$$F_4 = I(\overrightarrow{SP} \times \vec{B})$$
$$F_4 = I\ SP.B\ sin\theta$$
$$F_4 = I\ b\ B\ sin\theta$$

the direction of force is perpendicular to the plane containing SP and B. It is directed upward direction as shown in figure.
As the force acting on the upper and lower sides are equal and opposite along the same line of action, they cancel each other. As the force acting on the sides QR and SP are equal and opposite along different lines of action they constitute a couple. Hence the rectangular coil experiences a torque.
Therefore, the magnitude of torque acting on the coil is

τ = *force x arm of couple*

= B I *l* *x* b cosθ = BI*l*bcosθ = BIAcosθ

If the rectangular coil having N number of turns then torque is given by

T = Nτ = BINAcosθ

Thus, the torque acting on a coil in a magnetic field depends on the number of turns, area of current loop, strength of current and magnetic field.